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NEGATIVE FACTORIAL MOMENTS OF POSITIVE RANDOM VARIABLES *)

by Yves LEPAGE **)

SUMMARY

This paper is concerned with the problem of finding a technique to obtain the expected value of functions of the form $[(X+A) \ldots (X+A+k-1)]^{-1}$ where X is a random variable with $X+A > 0$ almost surely and k is a positive integer. The technique is applied to the Poisson, geometric and negative binomial distributions.

1. INTRODUCTION

Let X be a random variable defined on a probability space (Ω, A, P) and suppose that $X+A > 0$ a.s. (P). The purpose of this paper is to find a technique to obtain the expected value of functions of the random variable X, of the form

$$[(X+A) \ldots (X+A+k-1)]^{-1}$$

where k is a positive integer.

A technique of succesive integration of the factorial moment-generating function was suggested by Chao and Strawderman (see [1]) to obtain the expected value of functions of the random variable X, of the form

$$(X+A)^{-n}$$

where n is a non-negative integer. In section 2, this technique is modified to obtain the basic result and in section 3, it is applied to three special

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cases: the Poisson, geometric and negative binomial distributions. The utility of negative factorial moments arises specially in life testing problems.

2. THE BASIC RESULT

Define the factorial moment-generating function of X+A−1 by

$$g_1(t;A) = E(t^{X+A-1})$$

where $0 \le t \le 1$ and $A \in \mathbb{R}$.

Then, for k = 1,2,..., define

$$g_{k+1}(t;A) = \int_0^t g_k(u;A)\,du$$

where $0 \le t \le 1$ and $A \in \mathbb{R}$.

Theorem. *For* $0 \le t \le 1$ *and* $A \in \mathbb{R}$, *we have*

$$E\left[\frac{t^{X+A+k-1}}{(X+A)\ \ldots\ (X+A+k-1)}\right] = \int_0^t g_k(u;A)\,du$$

where k = 1,2,... .

Proof. Since X+A > 0, we have the following equality

$$\frac{t^{X+A+k-1}}{(X+A)\ \ldots\ (X+A+k-1)} = \int_0^t \left(\int_0^{t_k} \cdots \left(\int_0^{t_2} t_1^{X+A-1}dt_1\right)\ldots dt_{k-1}\right)dt_k .$$

Consequently, by taking the expectation on each side of the equality, we obtain the desired result. □

Corollary.

$$E\left[[(X+A)\ \ldots\ (X+A+k-1)]^{-1}\right] = \int_0^1 g_k(u;A)\,du .$$

Proof. The result is obtained by setting t = 1 in the preceeding theorem. □

3. APPLICATIONS

3.1. Poisson distribution

Let X be Poisson distributed with parameter λ. We know from Parzen (see [2], p.219), that

$$g_1(t;A) = t^{A-1}\, e^{\lambda t-\lambda}.$$

Suppose A = 1. Then, we have

$$g_2(t;1) = \int_0^t e^{\lambda u-\lambda} du = \frac{e^{-\lambda}}{\lambda}(e^{\lambda t}-1) .$$

Thus, after successive integrations, we obtain

$$g_k(t;1) = \frac{e^{-\lambda}}{\lambda}\left(\frac{e^{\lambda t}}{\lambda^{k-2}} - \sum_{r=0}^{k-2} \frac{t^r}{r!\,\lambda^{k-2-r}}\right) .$$

Consequently, for k = 2,3,..., it follows that

$$E([(X+1)\ \dots\ (X+k)]^{-1}) = \frac{1-e^{-\lambda}}{\lambda^k} - \sum_{r=0}^{k-2} \frac{e^{-\lambda}}{(r+1)!\,\lambda^{k-1-r}} .$$

In particular, we have

$$E([(X+1)(X+2)]^{-1}) = \frac{1}{\lambda^2}[1-e^{-\lambda}(1+\lambda)] .$$

3.2. Geometric distribution

Assume that X has a geometric distribution with parameter p ($p \in (0,1)$). We know, from Parzen (see [2], p.219), that

$$g_1(t;0) = \begin{cases} p/(1-tq) & \text{if } 0 < t \le 1, \\ 0 & \text{if } t = 0, \end{cases}$$

where q = 1-p. Thus, we have

$$g_2(t;0) = \int_0^t p/(1-up)du = - p/q \ \ell n(1-qt)$$

for $0 \le t \le 1$. Consequently, by carrying the integration, one can deduce that for k = 3,4,...,

$$g_{k+1}(t;0) = \frac{(-1)^k p}{(k-1)!q^k}(1-qt)^{k-1}\ell n(1-qt) + \sum_{r=1}^{k-1} a_r \frac{(-1)^{k-r+1} pt^r}{r!q^{k-r}}$$

where $a_1 = 1/(k-1)!$ and for r = 2,...,k-1,

$$a_r = \frac{1}{(k-r)!} + \frac{1}{(k-r-1)!}\sum_{s=2}^{r}\frac{1}{(k-s+1)} .$$

It follows that for k = 3,4,...,

$$E\left[[X(X+1) \ \dots \ (X+k-1)]^{-1}\right] = \frac{(-1)^k}{(k-1)!}\left(\frac{p}{q}\right)^k \ell n \ p + p\sum_{r=1}^{k-1} a_r \frac{(-1)^{k-r+1}}{r!q^{k-r}}$$

In particular, we have

$$E\left[[X(X+1)(X+2)(X+3)]^{-1}\right] = \frac{1}{6}\frac{p^4}{q^4}\ell n \ p + \frac{11}{36}\frac{p}{q} - \frac{5}{12}\frac{p}{q^2} + \frac{1}{6}\frac{p}{q^3},$$

and

$$E\left[[X(X+1)(X+2)(X+3)(X+4)]^{-1}\right] = -\frac{1}{24}\frac{p^5}{q^5}\ell n \ p + \frac{25}{288}\frac{p}{q} - \frac{13}{72}\frac{p}{q^2} + \frac{7}{48}\frac{p}{q^3} - \frac{1}{24}\frac{p}{q^4} .$$

3.3. Negative binomial distribution

Let X have a negative binomial distribution with parameters r (a positive integer) and p ($p \in (0,1)$). From Parzen (see [2], p.219), we have that

$$g_1(t,1) = p^r/(1-qt)^r, \quad 0 \le t \le 1,$$

where $q = 1-p$. Hence,

$$g_2(t,1) = \int_0^1 p^r/(1-qu)^r \, du =$$

$$= \frac{p^r}{(-r+1)(1-qt)^r(-q)} - \frac{p^r}{(-r+1)(-q)}$$

if $r \geq 2$. One can then deduce that if $r = k+1, k+2, \ldots$,

$$g_{k+1}(t,1) = \frac{p^r}{(r-1)(r-2)\ldots(r-k)(1-qt)^{r-k}} - \sum_{s=0}^{k-1} \frac{p^r t^s}{s!(r-1)\ldots(r-k+s)q^{k-s}} .$$

Thus, it follows that if $r = k+1, k+2, \ldots$,

$$E\left[[(X+1) \ \ldots \ (X+k)]^{-1}\right] =$$

$$= p^r \left[\frac{1}{(r-1)\ldots(r-k)p^{r-k}q^k} - \sum_{s=0}^{k-1} \frac{1}{s!(r-1)\ldots(r-k+s)q^{k-s}} \right] .$$

In particular, we find

$$E\left[[(X+1)(X+2)(X+3)(X+4)]^{-1}\right] =$$

$$= p^r \left[\frac{1}{(r-1)(r-2)(r-3)(r-4)p^{r-4}q^4} - \frac{1}{(r-1)(r-2)(r-3)(r-4)q^4} + \right.$$

$$\left. - \frac{1}{(r-1)(r-2)(r-3)q^3} - \frac{1}{2(r-1)(r-2)q^2} - \frac{1}{6(r-1)q} \right]$$

if $r = 5, 6, \ldots$.

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[2] Parzen, E., *Modern Probability Theory and Its Applications*. John Wiley and Sons, Inc., New York, 1960.